Formal Theories of Predication
Part II: Lecture Four
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1 Homogeneous Stratification

Something very much like the second-order predicate logic with nominalized predicates as abstract singular terms that we have described in the previous lecture was the form of logical realism, or ontological logicism, that Russell informally had in mind in his 1903 *Principles of Mathematics*. After discovering his paradox and trying many ways to resolve it, Russell settled on his theory of ramified types. Unfortunately, this theory, as well as the later theory of simple types, sets limits on what is meaningful or significant in natural language, and for that reason, not without some justification, it has been criticized.

As a way to avoid the paradox, what the theory of simple logical types does is divide the predicate expressions (and their corresponding abstract singular terms) of the second-order logic described in the previous lecture into a hierarchy of different types, and then it imposes a grammatical constraint that nominalized predicates can occur as argument- or subject-expressions only of predicates of higher types. This purely grammatical constraint excludes from the theory expressions of the form $F(F)$, or $F([\lambda x F(x)])$, as well as their negations, $\neg F(F)$, or $\neg F([\lambda x F(x)])$, which are just the types of expressions needed to generate Russell's paradox.[1] This was all that Russell needed to avoid his paradox of predication; but, as a way to avoid the so-called semantical paradoxes—such as that of the liar—Russell also divided the predicates on each level of the hierarchy of types into a ramified hierarchy of orders. The *simple* theory of types, which is

[1] Instead of the λ-operator, Russell used a cap-notation, $\varphi(\hat{x})$, to represent a property expression as an abstract singular term. In Russell's notation, type theory excluded formulas of the form $\varphi(\varphi(\hat{x}))$ and $\neg\varphi(\varphi(\hat{x}))$, which in our notation correspond to $F([\lambda x F(x)])$ and $\neg F([\lambda x F(x)])$. In the logic described in the preceding section,

$$F = [\lambda x F(x)]$$

is assumed to be valid.

all that is needed to avoid Russell's paradox, is based only on the first hierarchy, whereas the *ramified* theory of types is based on both.

These grammatical constraints are undesirable, we want to emphasize, because they exclude as meaningless many expressions that are not only grammatically correct in natural language but also intuitively meaningful, and that sometimes even true. Fortunately, it turns out, the logical insights behind these constraints can be retained while mitigating the constraints themselves. In particular, we need only one constraint on λ-abstracts, namely that they be restricted to those that are homogeneously stratified in a metalinguistic sense. Here, by a metalinguistic characterization, we mean one that applies only in the metalanguage and not in the object language as a distinction between types of predicates.

Retaining the same logical syntax that we described in the previous section, we say that a formula or λ-abstract φ is homogeneously stratified (or just h-stratified) if, and only if there is an assignment (in the metalanguage) t of natural numbers to the terms and predicate expressions occurring in φ (including φ itself if it is a λ-abstract) such that

- (1) for all terms a, b, if $(a = b)$ occurs in φ, then $t(a) = t(b)$;
- (2) for all $n \geq 1$, all n-place predicate expressions π, and all terms $a_1, ..., a_n$, if $\pi(a_1, ..., a_n)$ is a formula occurring in φ, then (i) $t(a_i) = t(a_j)$, for $1 \leq i, j \leq n$, and (ii) $t(\pi) = t(a_1) + 1$;
- (3) for $n \geq 1$, all individual variables $x_1, ..., x_n$, and formulas χ, if $[\lambda x_1...x_n\chi]$ occurs in φ, then (i) $t(x_i) = t(x_j)$, for $1 \leq i, j \leq n$, and (ii) $t([\lambda x_1...x_n\chi]) = t(x_1) + 1$; and
- (4) for all formulas χ, if $[\chi]$ (i.e., $[\lambda\chi]$) occurs in φ and $a_1, ..., a_k$ are all of the terms or predicates occurring in χ, then $t([\chi]) \geq \max[t(a_1), ..., t(a_{k)}]$.

The one constraint we need to impose to retain a consistent version of Russell's earlier 1903 logical realism is that, to be grammatically well-formed, all λ-abstracts must be homogeneously stratified (in the metalinguistic sense defined). This means that formulas of the form $F(F)$ and $\neg F(F)$ are still grammatically meaningful even though they are not h-stratified. Formulas of the form

$$F([\lambda x F(x)]), \quad \neg F([\lambda x F(x)]),$$
$$[\lambda x F(x)]([\lambda x F(x)]), \quad \neg[\lambda x F(x)]([\lambda x F(x)])$$

are also grammatically well-formed so long as the λ-abstracts in these formulas are h-stratified. On the other hand, note that the complex predicate that is involved in Russell's paradox, namely,

$$[\lambda x(\exists G)(x = G \wedge \neg G(x))],$$

is not h-stratified, because, x and G must be assigned the same number (level) for their occurrence in $x = G$, whereas G must also be assigned the successor of what x is assigned for their occurrence in $\neg G(x)$.

The comprehension principle $(\mathbf{CP}^*_\lambda)$ and the second-order logic of the previous section can be retained in its entirety, with the one restriction that the λ-abstracts that occur in the formulas of this logic must all be h-stratified. Because of this one restriction we will refer to the system as $\lambda\mathbf{HST}^*$.

Finally, let us note that not only is Russell's paradox blocked in $\lambda\mathbf{HST}^*$, but so are other logical paradoxes as well. As we have shown elsewhere, $\lambda\mathbf{HST}^*$ is consistent relative to Zermelo set theory and equiconsistent with the simple theory of logical types.[2] Also, if we were to add to $\lambda\mathbf{HST}^*$ the following axiom of extensionality,

$$(\forall x_1)...(\forall x_n)[F(x_1,...x_{,n}) \leftrightarrow G(x_1,...x_{,n})] \rightarrow F = G, \qquad (\mathbf{Ext}^*)$$

or, equivalently, because

$$F^n = [\lambda x_1...x_n F(x_1,...x_{,n})],$$

is valid in $\lambda\mathbf{HST}^*$ (and in fact is taken as an axiom),

$$\begin{aligned}(\forall x_1)...(\forall x_n)[F(x_1,...x_{,n}) \quad &\leftrightarrow \quad G(x_1,...x_{,n})] \rightarrow \qquad (\mathbf{Ext}^*)\\ [\lambda x_1...x_n F(x_1,...x_{,n})] \quad &= \quad [\lambda x_1...x_n G(x_1,...x_{,n})]\end{aligned}$$

then the result is equiconsistent with the set theory known as **NFU** (New foundations with Urelements) as well.[3]

Metatheorem: λHST* is consistent relative to Zermelo set theory; and it is equiconsistent with the theory of simple logical types. λHST*+ (Ext*) is equiconsistent with the set theory NFU.

2 Frege's Logic Reconstructed

Gottlob Frege's form of logical realism as described in his *Grundgesetze* was also a second-order predicate logic with nominalized predicates as abstract singular terms, and it too was subject to Russell's paradox.[4] But Frege also had a hierarchy of universals implicit in his logic, except that he assumed that all higher levels of his hierarchy beyond the second could be reflected downward into the second level, which in turn was reflected in the first level of objects, which is implicitly the situation that is represented in $\lambda\mathbf{HST}^*$.[5] In this respect,

[2] Cf. Cocchiarella 1986.

[3] See Holmes 1999 for a development of NFU.

[4] See Cocchiarella 1987, chapter 2, section 6, and chapter 4, section 3, for a detailed defense of the claim that Frege's extensional logic of *Wertverläufe* is really a logic of nominalized predicates.

[5] This reflection downward in Frege's logic is what I have referred to elsewhere as Frege's double correlation thesis. See Cocchiarella 1987, chapter two, section9, for a discussion of this part of Frege's logic. Implicit in this reflection is a rejection of Cantor's power-set theorem as applied to Frege's logic of extensions.

λ**HST*** can also be used as a consistent reconstruction of Frege's form of logical realism.

Unlike Russell, however, Frege did not assume that what a nominalized predicate denotes as an abstract singular term is the same universal that the predicate stands for in its predicative role. In addition, Frege's universals, which he called concepts (*Begriffe*) and relations, but which we will call properties and relations instead, have an unsaturated nature, and this unsaturated nature precludes the properties and relations of Frege's ontology from being objects.[6] For this reason, Frege's universals cannot be what nominalized predicates denote as abstract singular terms. In other words, in Frege's ontology what a predicate stands for in it predicative role is not what a nominalized predicate denotes as an abstract singular term.

Why then have nominalized predicates at all? In Frege's ontology it was not just to explain an important feature of natural language. Rather, it was a matter of "how we are to conceive of logical objects," and numbers in particular.[7] "By what means," Frege noted, "are we justified in recognizing numbers as objects?" The answer, for Frege, was that we apprehend logical objects as the extensions of properties and relations (or concepts), and it is through the process of nominalization that we are able to achieve this. Here, it is the logical notion of a class as the extension of a property, or concept, that is involved, and not the mathematical notion of a set.

> Unlike a set, which has its being in its members, a class in the logical sense has its being in the property, or concept, whose extension it is.[8]

Now it was Frege's commitment to an extensional logic that led him to take classes as the objects denoted by nominalized predicates. A class, after all, is the extension of a predicate as well as of the property or concept that the predicate stands for, and it was in terms of classes and classes of classes that Frege proposed to construct the natural numbers. That, in fact, is the basis of his ontological logicism. Notationally, Frege used for this purpose the spiritus lenis, or smooth-breathing symbol, as a variable-binding operator. Thus, given a formula φ and a variable ε, applying the spiritus lenis resulted in an expression of the form $\acute{\epsilon}\varphi(\epsilon)$, which Frege took to be a nominalized form of the predicate represented by $\varphi(\varepsilon)$. The smooth-breathing operator functioned in Frege's logic in much the same way as the λ-operator does in the logics we have described, and for that reason we will continue to use the λ-operator here instead.

Given Frege's commitment to an extensional logic, then it is not just λ**HST*** that we should take as a consistent reconstruction of his logic, but λ**HST***+ **(Ext*****)**, which, as already noted, is equiconsistent with the set theory **NFU** and consistent relative to Zermelo set theory.

The extensionality axiom, **(Ext*****)**, incidentally, is one direction of Frege's well-known **Axiom V**, which was critical to the way Russell's paradox was

[6] Frege sometimes also described concepts as properties (*Eigenshaften*) as well.
[7] Frege 1893, p. 143.
[8] Cf. Frege 1979, p. 183.

proved in Frege's logic. This direction was called **Basic Law Vb**. The other direction, **Basic Law Va**, is actually an instance of Leibniz's law in λ**HST***. That is, by **(LL*****)**, Frege's **Basic Law Va**,

$$F = G \to (\forall x_1)...(\forall x_n)[F(x_1, ...x_{,n}) \leftrightarrow G(x_1, ...x_{,n})]$$

is provable in λ**HST***, independently of **(Ext*****)**, which was Frege's **Basic Law Vb**. Given the consistency of λ**HST***+ **(Ext*****)** (relative to Zermelo set theory), which includes Leibniz's law, **(LL*****)**, as a theorem schema, it is not Frege's **Basic Law V** that was the problem so much as the way his hierarchy of universals was reflected downward into the first- and second-order levels. This was because Frege had heterogeneous, and not just homogeneous, relations in his logic, including heterogenous relations between universals and objects, such as that of predication, and these were included as part of the reflection downward of his hierarchy.[9] The hierarchy consistently represented in λ**HST***, on the hand, consists only of homogeneous relations.

The representation of heterogenous relations can be retained, however, by turning to an alternative reconstruction of Frege's logic that is closely related to λ**HST***. This alternative involves replacing the standard first-order logic that is part of λ**HST*** with a logic that is free of existential presuppositions regarding singular terms, including nominalized predicates such as that corresponding to the complex predicate involved in Russell's paradox.

Now it is significant that in an appendix to his *Grundgesetze* Frege considered resolving Russell's paradox by allowing that "there are cases where an unexceptional concept has no extension".[10] Here, by an "unexceptional concept" Frege had the rather exceptional Russell property (or concept) in mind. After all, what is exceptional about the Russell property in Frege's logic is that it leads to a contradiction, unless, that is, we allow that it has no extension. But allowing that the Russell property has no extension in Frege's logic requires allowing the nominalized form of the Russell predicate to denote nothing. That is, it requires a shift from standard first-order logic to a logic free of existential presuppositions regarding singular terms, including especially nominalized predicates.

In fact, this strategy works. By adopting a free first-order logic and yet retaining the unrestricted comprehension principle **(CP**$^*_\lambda$**)**, all that follows by the argument for Russell's paradox is that there is no object corresponding to the Russell property, i.e.,

$$\neg(\exists y)([\lambda x(\exists G)(x = G \wedge \neg G(x))] = y)$$

is provable, even though, by **(CP**$^*_\lambda$**)**, the Russell property "exists" as a property (or concept); that is, even though

$$(\exists F)([\lambda x(\exists G)(x = G \wedge \neg G(x))] = F)$$

[9] See Cocchiarella 1987, section 9, for a more detailed discussion of this point.
[10] Frege 1893, p. 128.

is also provable. Because the first-order logic is now a "free" logic, the original rule of λ-conversion must be modified as follows:

$$[\lambda x_1...x_n\varphi](a_1,...,a_n) \leftrightarrow (\exists x_1)...(\exists x_n)(a_1 = x_1 \wedge ... \wedge \ a_n = x_n \wedge \varphi), \quad (\exists/\lambda\text{-Conv}^*)$$

where, for all $i, j \leq n$, x_i doe not occur in a_j.

Revised in this way, our original second-order logic with nominalized predicates can be easily shown to be consistent. But that is because, without any further assumptions, we can no longer prove that any property or relation has an extension. That is, because the logic is free of existential presuppositions, all nominalized predicates might be denotationless, a position that a nominalist might well adopt.

But in Frege's ontological logicism some properties and relations must have extensions, and, indeed, it would seem to be appropriate to assume that all of the properties and relations that can be represented in $\lambda\mathbf{HST}^*$ have extensions in this alternative logic. That in fact is exactly what we allow in our alternative reconstruction of Frege's logic.

The added assumption can be stipulated in the form of an axiom schema. But to do so we need to first define the key notion of when an expression of our logical grammar can be said to be *bound to objects*.

Definition: If ξ is a meaningful expression of our logical grammar, i.e., $\xi \in \mathbf{ME}_n$, for some natural number n,then ξ is bound to objects if, and only if, for all predicate variables F, and all formulas $\varphi \in \mathbf{ME}_1$, if $(\forall F)\varphi$ is a formula occurring in ξ, then for some object variable x and some formula ψ, φ is the formula $[(\exists x)(F = x) \to \psi]$.

To be bound to objects, in other words, every predicate quantifier occurring in an expression ξ must refer only to those properties and relations (or concepts) that have objects corresponding to them, which in Frege's logic are classes as the extensions of the properties or relations in question. The axiom schema we need then is given as follows:

$$(\exists y)(a_1 = y) \wedge ... \wedge (\exists y)(a_k = y) \to (\exists y)([\lambda x_1...x_n\varphi] = y), \qquad (\exists/\mathbf{HSCP}^*_\lambda)$$

where,

- (1) $[\lambda x_1...x_n\varphi]$ is h-stratified,
- (2) φ is bound to objects,
- (3) y is an object variable not occurring in φ, and
- (4) $a_1,...,a_k$ are all of the object or predicate variables or non-logical constants occurring free in $[\lambda x_1...x_n\varphi]$.[11]

[11] We understand the conditional posited in this axiom schema to reduce to just the consequent if $k = 0$, i.e., if $[\lambda x_1...x_n\varphi]$ contains no free variables or nonlogical constants.

Because of it close similarity to our first reconstructed system, λ**HST***, we will refer to this alternative logic as $\mathbf{HST}^*_\lambda$.[12] As we have shown elsewhere, $\mathbf{HST}^*_\lambda$ is equiconsistent with λHST*, and therefore with the theory of simple types as well. It is of course also consistent relative to Zermelo set theory.

Finally, let us note that although $\mathbf{HST}^*_\lambda$ + **(Ext*****)** can be taken as a reconstruction of Frege's logic and ontology, it cannot also be taken as a reconstruction of Russell's early (1903) ontology, even without the extensionality axiom, **(Ext*****)**. This is because Russell rejected Frege's notion of unsaturatedness and took nominalized predicates to denote as singular terms the same concepts and relations they stand for as predicates. In other words, unlike Frege, Russell cannot allow that some predicates stand for properties and relations (or concepts), but that as singular terms their nominalizations denote nothing. Of course, we do have the logical system λHST*, which can be taken as a reconstruction of Russell's early ontological framework.

3 Conceptual Intensional Realism

In conceptualism, predicable concepts are cognitive capacities that underlie our rule-following abilities in the use of the predicate expressions of natural language, and as such concepts determine the truth conditions of that use. As capacities that can be exercised by different persons at the same time, as well as by the same person at different times, concepts cannot be objects, e.g., ideas or mental images as particular mental occurrences.

> In other words, as intersubjectively realizable cognitive capacities, concepts are objective and not merely subjective entities.

Moreover, as essential components of predication in language and thought, concepts as cognitive capacities have an unsaturated nature, and it is this unsaturated nature that is the basis of predication in language and thought. In particular, it is the exercise of a predicable concept in a speech or mental act that informs that act with a predicable nature, a nature by means of which we characterize and relate objects in various ways.

The unsaturatedness of a concept as a cognitive capacity is not the same as the unsaturatedness of a universal in Frege's ontology. For Frege, a property or relation is really a function from objects to truth values, and it is part of the nature of every function, according to Frege, even those from numbers to numbers, to be unsaturated. Predication, in other words, is reduced to functionality in Frege's ontology.

> In conceptualism it is predication that is more fundamental than functionality.

[12] For a detailed account of all of the axioms and properties of these systems see Cocchiarella 1986, chapter V.

We understand what it means to say that a function assigns truth values to objects, after all, only by knowing what it means to predicate concepts, or properties and relations, of objects. The unsaturated nature of a concept is not that of a function, but of a cognitive capacity that could be exercised by different people at the same time as well as by the same person at different times, and it might in fact not even be exercised ever at all. When such a capacity is exercised , however, what results is not a truth value, whatever sort of entity that might be, but a mental event, and if expressed overtly in language then a speech-act event as well.

Now we should note that conceptual thought consists not just of predicable concepts, but of referential and other types of concepts as well. Referential concepts, for example, are cognitive capacities that underlie our ability to refer (or really to purport to refer) to objects, and, as such, they too have an unsaturated cognitive structure. More importantly, referential concepts have a structure that is complementary to that of predicable concepts, so that each, when exercised or applied jointly in a judgment, mutually saturates the other, resulting thereby in an act (event) that is informed with a referential and a predicable nature. As we will explain in a later lecture, it is the complementarity of predicable and referential concepts as unsaturated cognitive structures that is the basis of the unity of our thoughts and speech acts, and which explains why a transcendental subjectivity need not be assumed as the basis of this unity. In this lecture, however, we will restrict ourselves to the predicable concepts that are the counterparts of the universals of logical realism.

If predicable concepts are unsaturated cognitive structures, then what point is there in having a logic of nominalized predicates as abstract singular terms? The point is the same as it was for Frege, namely to account for the ontology of the natural numbers as logical objects, and to explain the significance of nominalized predicates in natural language, including especially complex forms of predication containing infinitives, gerunds, and other abstract nouns. There is a difference, however, in that whereas Frege was committed to an extensional logic, conceptualism, once it admits abstract objects into its ontology, is committed to an intensional logic. Instead of denoting the extensions of concepts, in other words, nominalized predicates in conceptualism denote the intensional contents of concepts, which is why we refer to this extension of conceptualism as conceptual intensional realism.

> In conceptual intensional realism nominalized predicates denote the intensional contents of concepts.

Now by the intensional content of a predicable concept we understand an abstract intensional object corresponding to the truth conditions determined by the different possible applications of that concept, i.e., the conditions under which objects can be said to fall under the concept in any possible context of use, including fictional contexts. Of course, there are some predicable concepts, such as that represented by the Russell predicate,

$$[\lambda x(\exists G)(x = G \wedge \neg G(x))],$$

that determine truth conditions corresponding to which, logically, there can be no corresponding abstract object, on pain of contradiction. This does not mean that such a predicable concept does not determine truth conditions and therefore does not have intensional content; rather, it means only that such a content cannot be "object"-ified, i.e., there cannot be an abstract object corresponding to the content of that concept, the way there are for the contents of other concepts. The real lesson of Russell's paradox is that some rather exceptionable, "impredicatively" constructed concepts determine truth conditions that logically cannot be "object"-ified, whereas most predicable concepts are unexceptionable in this way.

In conceptual Platonism, incidentally, the abstract object corresponding to the intensional content of a predicable concept is a Platonic Form, which traditionally has also been called a property or relation, which we can allow as well in conceptual realism so long as we do not confuse these properties and relations with the natural properties and relations of conceptual natural realism. There is an important ontological difference between conceptual Platonism and conceptual intensional realism, however, despite the similarity of both to logical realism.

Unlike logical realism, conceptual Platonism is an indirect and not a direct Platonism. That is, in conceptual Platonism, but not in logical realism, abstract objects are cognized only indirectly through the concepts whose correlates they are. This means that our representation of abstract objects is seen as a reflexive abstraction corresponding to the process of nominalization. In other words, even though abstract objects according to conceptual Platonism exist in a realm that transcends space, time and causality, and in that sense preexist the evolution of consciousness and the cognitive capacities that we exercise in thought and our use of language, nevertheless, from an epistemological point of view, no such entity can be cognized otherwise than as the correlate of a concept, i.e., as an abstract intensional object corresponding to the truth conditions determined by that concept.

In conceptual intensional realism, on the other hand, all abstract objects, despite having a certain amount of autonomy, are products of language and culture, and in that respect do not preexist the evolution of consciousness and the cognitive capacities that we exercise in language.[13] They are, in other words, evolutionary products of language and culture, and therefore depend ontologically on language and culture for their "existence," or being. Of course, abstract objects, especially numbers, are also an essential part of the means whereby further cultural development becomes possible. Nevertheless, as cultural products, the "existence" of abstract objects is primarily the result, and development of, the kind of reflexive abstraction that is represented by the process of nominalization. It was through the institutionalization of this process that abstract objects achieved a certain autonomy and, in time, became reified as objects. Abstract

[13] Cf. Popper and Eccles 1977, chap. P2, for a similar view. We should note, however, that although our view of abstract objects supports the Popper-Eccles interactionist theory of mind, it does not also depend on that theory.

objects do not exist in a Platonic realm outside of space, time, and causality, on our interpretation, but are in fact the result, in effect, of an ontological projection inherent in the development and institutionalization in language of the process of nominalization.

The fundamental insight into the nature of *abstract objects* according to conceptual intensional realism is that we are able to grasp and have knowledge of such objects as the "object"-ified truth conditions of the concepts whose contents they are, i.e., as the object correlates of those concepts. This "object"-ification of truth conditions is realized, moreover, through a kind of reflexive abstraction in which we attempt to represent what is not an object—in particular an unsaturated cognitive structure underlying our use of a predicate expression—as if it were an object. In language this reflexive abstraction is institutionalized in the rule-based linguistic process of nominalization.

4 Actualism versus Possibilism Revisited

We have seen how actualism and possibilism can be distinguished from one another in tense logic and with respect to the logic of the temporal modalities of Aristotle and Diodorus. A deeper analysis can be given, however, in terms of the second-order theories of predication for both logical and conceptual realism. We could, for this purpose, restrict ourselves to second-order tense logic, which would be appropriate for conceptual realism because as forms of conceptual activity, thought and communication are inextricably temporal phenomena. Tense logic, in other words, is implicitly assumed as a fundamental part of the formal ontology of conceptual realism. A causal or natural modality is also a fundamental part of conceptual natural realism, and with this modality comes the distinction between what is actual and what is possible in nature. That is, with a causal or natural modality we have an even sharper distinction between actualism and possibilism.

Logical realism does not reject tense logic and the temporal modalities; nor does it reject a causal or natural modality. But it does assume a still stronger modal notion that need not be a fundamental part of conceptual realism, namely a metaphysical modality. This is a difficult notion to explicate, however, because we do not have precise criteria by which to determine the metaphysical conditions appropriate for logical realism in the semantics of a logic of metaphysical necessity and possibility. What we need is some way by which to understand the key notion of a metaphysically possible world—if such a notion is in fact really different from what is possible in nature.

In the ontology of logical atomism, which is a form of natural realism, we do have a notion of a possible world that is sharp and clear, because every possible world in logical atomism is made up of the same atomic states of affairs, and hence the same simple material objects and properties and relations, as those that make up the actual world, the only difference being between those states of affairs that obtain in a given world and those that do not. Different possible worlds in logical atomism, in other words, amount to different permutations of being-the-case and not-being-the-case of the same atomic states of affairs, and hence the same simple material objects and properties and relations, that make up the actual world. As a result, in such an ontology we have a precise notion of what it means to refer to "all possible worlds," and dually to "some possible world", which are the key notions characterizing how necessity and possibility are understood in this ontology. But then, the notion of a possible world in logical atomism is not at all suitable for logical realism with its commitment to a realm of abstract objects and a network of complex relations between them.

> Question: What notion of a metaphysically possible world is appropriate for logical realism?[14]

We will not attempt to answer this question here, but we will assume that some explication can in principle be given, and that all metaphysically possible worlds are equally accessible from other metaphysically possible worlds, so that the modal logic for metaphysical necessity contains at least the laws of **S5**. We will also assume—but only for the purpose of finding a difference that we will explain later—that the notions of metaphysical necessity and possibility correspond, at least roughly, to the equally difficult notions of conceptual necessity and possibility, even though it is not clear that these modal notions are an essential part of conceptual realism as well. In any case, we will use $\Box$ and $\Diamond$ as modal operators for both metaphysical and conceptual necessity and possibility. We will also speak of possibilism and actualism as a distinction applicable to both logical and conceptual realism. The second-order modal predicate logics with nominalized predicates as abstract singular terms that result from this addition are called $\Box\lambda\mathbf{HST}^*$ and $\mathbf{HST}^*_{\lambda\Box}$.[15]

5 Existence-Entailing Properties and Relations

We assume that the logic of possible objects described in our second lecture has been modified in accordance with the previous two sections, so that axiom **(A8)**, namely,

$$(\exists x)(a = x),$$

where x does not occur in a, has been changed to

$$(\forall x)(\exists y)(x = y),$$

where x, y are distinct object variables. In other words, the logic of possible objects is now a "free logic," just as the logic of actual objects is a free logic.

[14] What is involved in answering this question is determining the appropriate conditions for sets of models as set-theoretic counterparts of possible worlds in the semantic clauses for necessity and possibility. Allowing arbitrary sets of models as possible-world counterparts, where one can add or take away a model from such a set, does nothing by way explaining what is meant by metaphysical necessity.

[15] For a detailed axiomatization and discussion of $\Box\lambda\mathbf{HST}^*$ and $\mathbf{HST}^*_{\lambda\Box}$ see Cocchiarella 1986.

The difference between them is that whereas what is true of every possible object is therefore also true of every actual object, the converse does not also hold; that is,

$$(\forall x)\varphi \to (\forall^e x)\varphi$$

is a basic law, but the converse is not.

Strictly speaking, our so-called logic of possible objects now deals with more than merely possible objects, i.e., objects that actually exist in some possible world or other. Here, by *existence*—or for emphasis, *actual existence*—we mean *existence as a concrete object*, as opposed to the being of an abstract intensional object. Abstract intensional objects do not ever exist as actual objects in this sense even though they have *being* and are now taken as values of the bound object variables along with all possible objects. We can formulate this difference between actual existence and being as follows:

$$(\forall F^n)\neg E!(F). \qquad (\neg\mathbf{E!Abst})$$

Now it is noteworthy that in actualism there is no distinction between being and existence; that is, actualism is committed to there being only actually existing objects. Actualism can allow for nominalized predicates, but only as vacuous, i.e., nondenoting, singular terms. In that case, actualism will validate something like ¬**E!Abst**, but expressed in terms of an actualist predicate quantifier as described below. In possibilism, or rather in what we are now calling possibilism, there is a categorial distinction between being and possible existence, which is expressed in part by the validity of ¬**E!Abst**. In fact, this particular categorial distinction is one of the many—perhaps indeterminately many—conditions needed to characterize a metaphysically possible world. For convenience, however, we will continue to speak of the first-order logic of being, which includes possible existence, as simply the logic of possibilism.

The actualist quantifiers $\forall^e$ and $\exists^e$ when applied to predicate variables refer to those properties, concepts, or relations that only existing, actual objects can have, or fall under, in any given possible world. The following, accordingly, are valid theses of the second-order logic of possible objects:

$$(\forall^e F^n)\varphi \leftrightarrow (\forall F^n)(\Box(\forall x_1)...\forall x_n)[F(x_1,...,x_n) \to E!(x_1) \wedge ... \wedge E!(x_n)] \to \varphi),$$

and

$$(\exists^e F^n)\varphi \leftrightarrow (\exists F^n)(\Box(\forall x_1)...\forall x_n)[F(x_1,...,x_n) \to E!(x_1) \wedge ... \wedge E!(x_n)] \wedge \varphi).$$

We call the properties, concepts, and relations that only actual existing objects can fall under at any time in any possible world *existence-entailing* properties, concepts, and relations.

Now many properties, or concepts, and relations are such that only actual, existing objects can have, or fall under, them. In fact these are the more common properties or concepts that we ordinarily speak of in our commonsense framework. Thus, for example, an object cannot be red, or green, or blue, etc., at any time in a given possible world unless that objects exists in that world at that time. Similarly an object cannot be a pig or a horse at a time in a world unless it exists at that time in that world; nor, we should add, can there be a winged horse or a pig that flies unless it exists. Of course, in mythology there is a winged horse, namely Pegasus, and there could as well be a pig in fiction that flies. But that is not at all the same as actually being a winged horse or a pig that flies. Indeed, as far as fiction goes, there can even be a story in which there is an impossible object, such as a round square.[16] But a fictional or mythological horse is not a real, actually existing horse, and one of the tasks of a formal ontology is to account for the distinction between merely fictional and actually existing objects. Later, in a subsequent lecture we will explain how the ontology of fictional, or mythological, characters and entities such as the winged horse Pegasus, can be accounted for as intensional objects. For now it is important to distinguish actual existence from being and merely possible existence.

> The two main theses of actualism are:
> (1) quantificational reference to objects can be only to objects that actually existence, and
> (2) quantificational reference to properties, concepts, or relations can be only to those that "entail" existence in the above sense, i.e., the only properties, concepts or relations there are according to actualism are those that only actually existing objects can have or fall under.

What this means is that in actualism the quantifiers $\forall^e$ and $\exists^e$, must be taken as primitive symbols when applied to object or predicate variables. The following, moreover, is a basic theorem of actualism.

$$(\forall^e F^n)[F(x_1, ..., x_n) \rightarrow E!(x_1) \wedge ... \wedge E!(x_n)].$$

In regard to the concept of existence, note that the statement that every object exists, i.e., $(\forall^e x)E!(x)$, is a valid thesis of actualism. In possibilism, however, the same statement is false, and in fact, given the being of abstract intensional objects, none of which ever exist as actual objects, it is logically false that every object exists; that is, $\neg(\forall x)E!(x)$, is a valid thesis of possibilism as we understand it here.[17] What is true in both possibilism and actualism, on the other hand, is the thesis that *to exist* is *to possess, or fall under, an existence-entailing property or concept*; that is,

$$E!(x) \leftrightarrow (\exists^e F)F(x)$$

is valid in both actualism and possibilism.

[16] See, e.g., the story *Romeo and Juliet in Flatland* in Cocchiarella 1996, § 7.

[17] The universal and null concepts $[\lambda x(x = x)]$ and $[\lambda x(x \neq x)]$ are h-stratified and therefore provably have intensional objects as their object-correlates in both λHST* and HST^*_λ.

In possibilism, in fact, by taking quantification over existence-entailing properties or concepts as primitive, or basic, we can define existence as follows:

$$E! =_{df} [\lambda x(\exists^e F)F(x)].$$

What this definition indicates is that the concept of existence is an "impredicative" concept. In other words, because existence is itself an existence-entailing concept—i.e., if a thing exists, then it exists—then, the concept of existence is formed or constructed in terms of a totality to which it belongs. This, in fact, is why according to conceptual realism the concept of existence is so different from ordinary existence-entailing concepts, such as being red, or green, a horse, a tree, etc.

Identity, incidentally, coincides with indiscernibility in both possibilism and actualism. In possibilism this thesis that identity coincides with indiscernibility is formulated as follows:

$$(\forall x)(\forall y)(x = y \leftrightarrow (\forall F)[F(x) \leftrightarrow F(y)]).$$

In actualism, the thesis is stated in a more restricted way, namely, as

$$(\forall^e x)(\forall^e y)(x = y \leftrightarrow (\forall^e F)[F(x) \leftrightarrow F(y)]).$$

There is another difference on this matter if we formulate the thesis without the initial quantifiers. That is, whereas in possibilism, the following

$$x = y \leftrightarrow (\forall F)[F(x) \leftrightarrow F(y)]$$

is valid, in actualism the related formula

$$x = y \leftrightarrow (\forall^e F)[F(x) \leftrightarrow F(y)]$$

is actually false in the right-to-left direction when neither x nor y exist. In other words, if x and y do not exist, then they vacuously fall under all the same existence-entailing concepts, namely none; and yet it does not follow that that $x = y$. Neither Pegasus nor Bellerophon actually exist, and yet it would be false to conclude that therefore Pegasus is Bellerophon. In other words,

$$\neg E!(x) \wedge \neg E!(y) \rightarrow (\forall^e F)[F(x) \leftrightarrow F(y)]$$

is valid in both possibilism and actualism even when $x \neq y$.

The principle of universal instantiation for actualist quantifiers can be formulated as follows in possibilism:

$$(\exists^e F^n)([\lambda x_1...x_n\varphi] = F) \rightarrow ((\forall^e G)\psi \rightarrow \psi[\varphi/G(x_1, ..., x_n)]). \qquad (\exists/\mathbf{UI}_2^e)$$

In actualism, where λ-abstracts cannot be nominalized, the formulation, as follows, is somewhat more complex regarding its antecedent condition:

$$(\exists^e F^n)\Box(\forall x_1)\Box(\forall x_2)...\Box(\forall x_n)\Box[F(x_1, ..., x_n) \leftrightarrow \varphi] \rightarrow ((\forall^e G)\psi \rightarrow \psi[\varphi/G(x_1, ..., x_n)]).$$

The comprehension principle for the actualist quantifiers is not as simple as the comprehension principle $(\mathbf{CP}^*_\lambda)$ for possibilism. It amounts to a kind of *Aussonderungsaxiom* for existence-entailing properties, concepts and relations.

$$(\forall^e G^k)(\exists^e F^n)([\lambda x_1...x_n(G(x_1,...,x_k)\wedge\varphi)]=F), \qquad (\mathbf{CP}^e_\lambda)$$

where $k \leq n$, and G^k and F^n are distinct predicate variables that do not occur in φ. In the monadic case, for example, what this principle states is that although we cannot expect every open formula φx to represent an existence-entailing property or concept, nevertheless conjoining φx with an existence-entailing property or concept $G(x)$ does represent a property or concept that entails existence. The formulation in actualism, where the nominalized λ-abstract is not allowed (as a denoting singular term), is again more complex, but we will forego those details here.[18] A simpler comprehension principle that is also valid in actualism is:

$$(\exists^e F^n)\Box(\forall^e x_1)...(\forall^e x_n)[F(x_1,...,x_n)\leftrightarrow\varphi].$$

Before concluding this section we should note that although many of our commonsense concepts are concepts that only existing objects can have, nevertheless there are some concepts, especially relational ones, that can hold between objects that do not exist in the same period of time in our world or even in the same world. All animals, for example, have ancestors whose lifespans do not overlap with their own, and yet they remain their ancestors. An acorn that we choose to crush under our feet will never grow into an oak tree in our world, and yet, as a matter of natural possibility, there is a world very much like ours in which we choose not to crush the acorn but leave it to grow into an oak tree. It is only a possible, and not an actual, oak tree in our world, but there is still relation between it and the acorn that we crushed, just as there is a ancestral relation between animals whose lifespans do not overlap.

6 Intensional Possible Worlds in Logical Realism

The actual world, according to conceptual realism, consists of physical objects and events of all sorts that are structured in terms of the laws of nature and the natural kinds of things there are in the world. There are intensional objects as well, to be sure, but these have being only as natural products of the evolution of consciousness, language and culture; and without language and thought they would not be at all. Possible worlds are like the actual world, moreover, i.e., they consist for the most part of physical objects and events structured in terms of the laws of nature.

In speaking of possible worlds here—and even of the actual world—we should be cautious to note that at least in conceptual realism possible worlds are not

[18]See Cocchiarella 1986, chapter III, §9, for a detailed axiomatization of actualism.

"objects," as when we speak of the various possible kinds of objects and events in the world. Possible worlds are not values of the bound objectual variables, in other words, and therefore we do not quantify over them as objects within the formal ontology of conceptual realism. It is true that we seem to quantify over possible worlds in a set-theoretic semantical metalanguage. But, to be precise, what we really quantify over are set-theoretical model structures that we call possible worlds. These set-theoretical model structures are not possible worlds in any appropriate metaphysical sense; rather, they are abstract mathematical objects that we use to represent the different possible situations described in the object-language of our formal ontology by means of modal operators. Unlike the model-theoretic "possible-world" parameters of the set-theoretical metalanguage, the modal operators of a formal ontology can be iterated and can occur within the scope of other occurrences of the same, or of a dual, operator. In addition, unlike the "external relations" expressed in the metalanguage between model-theoretic "possible worlds" and objects in the domains of those worlds, modal operators in the language of the formal ontology express "internal relations" between objects and the properties or concepts they fall under.

Possible worlds in logical realism are not all that much different, i.e., they are made up of physical objects and events, and, as in conceptual realism, they are not themselves *a kind of concrete object* in addition to the possible situations represented by the modal operators. But then, unlike the situation in conceptual realism, intensional objects have a mode of being in logical realism in a Platonic realm that is independent of the natural world, and even of whether or not there is a natural world. This Platonic realm is not only structured logically, as are the intensional objects of conceptual realism, but, because it does not depend upon our cognitive abilities, it goes beyond the latter in having abstract objects that are intensional counterparts of the possible worlds (models) that are supposedly quantified over in the metalanguage.

> That is, even though metaphysically possible worlds are not themselves *kinds of objects* in the formal ontology of logical realism, nevertheless there are intensional objects in this ontology that are the counterparts of metaphysically possible worlds—i.e., of the possible-world models of a set-theoretic metalanguage.
>
> In conceptual realism, on the other hand, there are no such intensional counterparts of possible worlds (models). This is because, in conceptual realism, but not in logical realism, all intensional objects are ontologically grounded in terms of the human capacity for thought and concept-formation.

Now there are at least two kinds of intensional objects in the ontology of logical realism that are counterparts of metaphysically possible worlds (models). For convenience, we will call both kinds *intensional possible worlds*. Our first such kind of intensional object is that of a proposition P that is both possible, i.e., $\Diamond P$, and maximal in the sense that for each proposition Q, either P entails Q or P entails $\neg Q$, where by "entailment" we mean necessary material implication, i.e., either $\Box(P \to Q)$ or $\Box(P \to \neg Q)$.[19] Where P is a possible-world counterpart in this sense, we read $\Box(P \to Q)$ as 'Q is true in P'. This notion can be specified by the following λ-abstract:

$$Poss\text{-}Wld_1 =_{df} [\lambda x(\exists P)(x = P \wedge \Diamond P \wedge (\forall Q)[\Box(P \to Q) \vee \Box(P \to \neg Q)])].$$

Note that this λ-abstract is a homogeneously stratified predicate. It follows, accordingly, that $Poss\text{-}Wld_1$ stands for a *property, or concept*, that is a value of the bound predicate variables of both $\Box\lambda\text{HST}^*$ and $\text{HST}^*_{\lambda\Box}$, and also that its nominalization denotes a value of the bound objectual variables in both of these systems.

Of course, the fact that $Poss\text{-}Wld_1$ is a well-formed predicate that stands for a property or concept does not mean that it must be true of anything, i.e., that there must be propositions that have this property, or fall under this concept. In fact, any proposition that falls under this concept as an intensional object has content that so far exceeds what is cognitively possible for humans to have as an object of thought that the "existence," or being, of such a proposition can in no sense be validated in conceptual realism. Of course, in logical realism, propositions are Platonic entities existing independently of the world and all forms of human cognition. In other words, the possible "existence," or being, of a proposition P such that $Poss\text{-}Wld_1(P)$, is not constrained in logical realism by what is cognitively possible for humans to have as an object of thought.

Nevertheless, we cannot prove that there are intensional possible worlds in this sense in either $\Box\lambda\mathbf{HST}^*$ and $\mathbf{HST}^*_{\lambda\Box}$, unless some axiom is added to that effect. One such axiom would be the following, which posits the "existence," or being, of a proposition corresponding to each metaphysically possible world (model) of the metalanguage.

$$\Box(\exists P)[Poss\text{-}Wld_1(P) \wedge P]. \qquad (\exists\mathbf{Wld}_1)$$

Of course, one immediate consequence of $(\exists\mathbf{Wld}_1)$ is that some intensional possible world now obtains, i.e., $(\exists P)[Poss\text{-}Wld_1(P) \wedge P]$, which we can refer to as "the intensional actual world."

A criterion of adequacy for this notion of a possible world is that it should yield the type of results we find in the set-theoretic semantics for modal logic. One such result is that a proposition is true, i.e., now obtains, if, and only if, it is true in the actual world. This result is in fact provable on the basis of $(\exists\mathbf{WLd}_1)$, i.e.,

$$Q \leftrightarrow (\exists P)[Poss\text{-}Wld_1(P) \wedge P \wedge \Box(P \to Q)],$$

is provable in both $\Box\lambda\mathbf{HST}^*$and $\mathbf{HST}^*_{\lambda\Box}$ if $(\exists\mathbf{Wld}_1)$ is added as an axiom.

From this another appropriate result follows; namely, that a proposition Q is possible, i.e., $\Diamond Q$, if it true in some possible world; that is,

$$\Diamond Q \leftrightarrow (\exists P)[Poss\text{-}Wld_1(P) \wedge \Box(P \to Q)],$$

[19]See Prior & Fine, 1977 for a discussion of this approach to intensional possible worlds.

is provable in both $\Box\lambda\mathbf{HST}^* + (\exists\mathbf{Wld}_1)$ and $\mathbf{HST}^*_{\lambda\Box} + (\exists\mathbf{Wld}_1)$.

Finally, another appropriate consequence is that if Q and Q' are true in all the same possible worlds (models), then they are necessarily equivalent; and, conversely, if they are necessarily equivalent, then they are true in all the same possible worlds; that is,

$$(\forall P)[Poss\text{-}Wld_1(P) \to (\Box(P \to Q) \leftrightarrow \Box[P \to Q'])] \leftrightarrow \Box[Q \leftrightarrow Q']$$

is provable in $\Box\lambda\mathbf{HST}^* + (\exists\mathbf{Wld}_1)$ and $\mathbf{HST}^*_{\lambda\Box} + (\exists\mathbf{Wld}_1)$. It does not follow, however, that propositions are identical if they are true in all of the same intensional possible worlds.

There is yet another notion of an intensional possible world that can have instances in logical realism but not in conceptual realism. This is the notion of an intensional possible world as a property in the sense of "the way things might have been." David Lewis, for example, claimed that possible worlds are "ways things might have been," but, curiously, according to Lewis the "ways that things might have been" are concrete objects, and not properties.[20] Robert Stalnaker, who is an actualist, pointed out, however, that "*the way things are* is a property or state of the world, [and] not the world itself," as Lewis would have it.[21] In other words, according to Stalnaker, "the ways things might have been" are properties. For actualist such as Stalnaker this means that there are possible worlds *qua* properties, but, except for the actual world, which is concrete, they are all uninstantiated properties. This is because, on Stalnaker's view, only concrete worlds could be instances of such properties, and as an actualist the only such concrete instance is the actual world. In other words, although there are possible worlds *qua* properties, according to Stalnaker, nevertheless there can be no possible worlds *qua* instances of those properties other than the actual world, because such instances would then be concrete and yet not actual, which is what actualism rejects. In logical realism, however, the situation is quite different.

> In the ontology of logical realism, the "ways things might have been" are properties, but they are not properties of metaphysically possible worlds as concrete objects. Rather, they are properties of propositions; in particular, they are properties of all and only the propositions that are true in the metaphysically possible world (model) in question.

Now a property of all and only the propositions that are true in a possible world (model) $\mathfrak{A}$ could not be an intensional counterpart of the world $\mathfrak{A}$ if the extension of that property were different in different possible worlds (models). Possible worlds are different, in other words, if the propositions true in those worlds are different. What is needed is a property of propositions that does not change its extension from possible world to possible world. We will call such a

[20] See Lewis 1973, p. 84. The relevant text is reprinted in Loux, 1979, p. 182.
[21] Stalnaker 1976, p. 228.

property a "rigid" property, which we "define" as follows:[22]

$$Rigid_n(F^n) \qquad : \qquad (\forall x_1)...(\forall x_n)[\Box F(x_1, ...x_n) \vee \Box\neg F(x_1, ...x_n)].$$

Note that because a rigid property will have the same extension in every possible world (model), the *extension* of that property can then be identified ontologically with the property itself.

The type of intensional possible world that is now under consideration is that of a rigid property (or "class") that holds in some metaphysically possible world (model) of all and only the propositions that are true in that world. This notion can be specified by a homogeneously stratified formula, which means that the property of being an intensional possible world in this sense can be defined by means of a λ-abstract as follows:

$$Poss\text{-}Wld_2 =_{df} [\lambda x(\exists G)(x = G \wedge Rigid_1(G) \wedge \Diamond(\forall y)[G(y) \leftrightarrow (\exists P)(y = P \wedge P)])].$$

Now, as with our first notion of an intensional possible world, the claim that there are possible worlds in this second sense is also not provable in either $\Box\lambda\mathbf{HST}^*$ or $\mathbf{HST}^*_{\lambda\Box}$, unless we add an assumption to that effect. One such assumption is the following, which says that there is such a possible-world property G, i.e., $Poss\text{-}Wld_2(G)$, that holds in any possible world (model) of all and only the propositions that are true in that world:

$$\Box(\exists G)(Poss\text{-}Wld_2(G) \wedge (\forall y)[G(y) \leftrightarrow True(y)]). \qquad (\exists\mathbf{Wld}_2)$$

Here by $True(y)$ we mean that y is a proposition that is the case, i.e.,

$$True =_{df} [\lambda y(\exists P)(y = P \wedge P)],$$

which, because the λ-abstract in question is homogeneously stratified, specifies a property, or concept, in both $\Box\lambda\mathbf{HST}^*$ and $\mathbf{HST}^*_{\lambda\Box}$.

Now it turns out that we do not have to assume either of the new axioms, **(**$\exists\mathbf{Wld}_1$**)** or **(**$\exists\mathbf{Wld}_2$**)**, to prove that there are intensional possible worlds in the formal ontology of logical realism in either of these two senses. Both, in fact, can be derived in the modal systems $\Box\lambda\mathbf{HST}^*$ and $\mathbf{HST}^*_{\lambda\Box}$ from what we will call *the principle of rigidity,* **(PR)***,* which stipulates that every property, or concept, F, is co-extensive in any metaphysically possible world (model) with a rigid property, i.e., a rigid property that can be taken as the extension of F (in that world).[23] We formulate the principle of rigidity as follows:

$$\Box(\forall F^n)(\exists G^n)(Rigid_n(G) \wedge (\forall x_1)...(\forall x_n)[F(x_1, ...x_n) \leftrightarrow G(x_1, ...x_n)]). \quad (\mathbf{PR})$$

The thesis that there is a rigid property corresponding to any given property is intuitively valid in logical realism where properties have a mode of being that

[22] Rigidity can be λ-defined as a predicate in $\Box\lambda$HST*. But in $\text{HST}^*_{\lambda\Box}$ it must be construed only as an abbreviation in the principle of rigidity described below.

[23] The idea of representing the extension of a property by means of a rigid property was first suggested by Richard Montague. A type-theoretical version of the thesis was used as a principle of "extensional comprehension" by Dan Gallin in his development of Montague's intensional logic. See Montague 1974, p.132, and Gallin 1975, p. 77.

is independent of our ability to conceive or form them as concepts. In conceptual realism, on the other hand, the thesis amounts to a "reducibility axiom" claiming that for any given concept or relation F we can construct a corresponding rigid concept or relation that in effect represents the extension of the concept F. Such a "reducibility axiom" is much too strong a thesis about our abilities for concept-formation.

That $(\exists\mathbf{Wld}_2)$ is derivable from **(PR)** follows from the fact that $True$ represents a property in these systems; that is,

$$(\exists G)(Rigid_2(G) \wedge (\forall y)[G(y) \leftrightarrow True(y)])$$

is provable on the basis of $(\mathbf{CP}^*_\lambda)$ in both $\Box\lambda\mathbf{HST}^*$+(**PR**) and $\mathbf{HST}^*_{\lambda\Box}$+(**PR**), and therefore, by the rule of necessitation and obvious theses of **S5** modal logic, it follows that $(\exists\mathbf{Wld}_2)$ is derivable from **(PR)**. A similar argument, which we will not go into here, shows that $(\exists\mathbf{Wld}_1)$ is also derivable from in $\Box\lambda\mathbf{HST}^*$+ (**PR**) and $\mathbf{HST}^*_{\lambda\Box}$ + (**PR**).

The fact that with **(PR)** we can prove in both of the theories of predication $\Box\lambda\mathbf{HST}^*$ and $\mathbf{HST}^*_{\lambda\Box}$ that there are intensional possible worlds in either the sense of $Poss\text{-}Wld_1$ and $Poss\text{-}Wld_2$ is significant in more than one respect. On the one hand it indicates the kind of ontological commitment that logical realism has as a modern form of Platonism. On the other hand, it also indicates a major kind of difference between logical realism and conceptual realism, because, unlike logical realism, the principle of rigidity is not valid in conceptual realism. What it claims about concept-formation is not cognitively realizable for humans. Nor can there be intensional possible worlds in the sense either of $Poss\text{-}Wld_1$ or $Poss\text{-}Wld_2$ in conceptual realism, because such intensional objects are not cognitively possible for human thought and concept-formation. Here, with the principle of rigidity and the notion of a proposition as the intensional counterpart of a possible world, we have clear distinction between logical realism as a modern form of Platonism and conceptual realism as a modern form of conceptualism, i.e., as a form of conceptual intensional realism. Also, what these differences indicate is that the notions of metaphysical necessity and possibility in logical realism are not the same as the notions of conceptual necessity and possibility in conceptual realism, at least not if conceptual possibility is grounded in what is cognitively realizable in human thought and concept-formation.

> Logical realism: The principle of rigidity, (PR), is valid, and therefore so are $(\exists\mathrm{Wld}_1)$ and $(\exists\mathrm{Wld}_2)$. That is, there are intensional possible worlds in logical realism in the sense of $Poss\text{-}Wld_1$ as well as of $Poss\text{-}Wld_2$.
>
> Conceptual realism: The principle of rigidity, (PR), is not valid, and there can be no intensional possible worlds in the sense of either $Poss\text{-}Wld_1$ or $Poss\text{-}Wld_2$, because such intensional objects exceed what is cognitively possible for human thought and concept-formation.
>
> Therefore, metaphysical necessity and possibility are the not the same as conceptual necessity and possibility.

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