On the (Onto)Logic of
Nominalized Predicates

1. Nominalization

Is there no difference between logical realism and holistic conceptualism as theories of predication, other than the fact that the latter presupposes a logic of "predicative" concepts as a proper part?

In fact, there is a difference once we consider the import of nominalized predicates and propositional forms as abstract singular terms in the wider context of modal predicate logic. The use of nominalized predicates as abstract singular terms is not only a part of our commonsense framework, but it is also central to how both logical realism and conceptual intensional realism provide an ontological foundation for the natural numbers and other parts of mathematics. This part of logical realism is sometimes called **ontological logicism**.

In Bertrand Russell's form of logical realism, e.g., universals are not just what predicates stand for in their role as predicates, but also what nominalized predicates denote as *abstract* singular terms.

Note: By *nominalization* we mean the transformation of a predicate phrase into an abstract noun, which is represented in logical syntax as an object term, i.e., the type of expression that can be substituted for first-order object variables. The following are some examples of predicate nominalizations:

$$\begin{array}{rcl} \text{is triangular} & \Rrightarrow & \text{triangularity} \\ \text{is wise} & \Rrightarrow & \text{wisdom} \\ \text{is just} & \Rrightarrow & \text{justice} \end{array}$$

It was Plato who first recognized the ontological significance of such a transformation and who built his ontology and his account of predication around it. In nominalism, of course, abstract nouns denote nothing.

In English we usually mark the transformation of a predicate into an abstract noun by adding such suffixes as '-ity', '-ness', or 'hood', as with 'triangularity', 'redness', and 'brotherhood'.

2. Logical Syntax

We do not need a special operator for this in logical syntax, however.

We need only delete the parentheses that are a part of a predicate variable or constant in its predicative role.

Thus, for a monadic predicate F we would have not only formulas such as $F(x)$, where F occurs in its predicative role, but also formulas such as $G(F)$, $R(x, F)$, where F occurs nominalized as an abstract object term.

Note: In $F(F)$ and $\neg F(F)$, F occurs both in it predicative role and as an abstract object term, though in no single occurrence can it occur both as a predicate and as a object term.

With nominalized predicates as abstract terms, we need to have complex predicates represented directly by using, e.g., the variable-binding λ-operator. Thus, where φ is a formula of whatever complexity and n is a natural number, we have a *complex predicate* of the form $[\lambda x_1 ... x_n \varphi](\ )$, which has parentheses accompanying it in its predicative role, but which are deleted when the complex predicate is nominalized.

Note: the nominalization of a formula φ has the form $[\lambda\varphi]$, which is read as 'that φ'. For convenience, we shall write '$[\varphi]$' for '$[\lambda\varphi]$'.

With λ-abstracts the grammar of our logical syntax is now more complicated of course. In particular, object terms and formulas must now be defined simultaneously. The definition and other details can be found in:

http://www.stoqnet.org/lat_notes.html, Course 50547: Elements of Formal Ontology, Lecture 4.

This new logical grammar contains what might be described as four of the essential parts of a theory of logical form: namely,

- The basic forms of predication, as in $F(x)$, $R(x, y)$, etc.;
- Propositional (sentential) connectives, e.g., $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$;
- Quantifiers that reach into predicate as well as subject (argument) positions;
- Nominalized predicates and propositional forms as abstract object terms.

These four components correspond to fundamental features of natural language, and each needs to be accounted for in any theory of logical form underlying natural language.

3. Russell's Paradox of Predication

As a framework for logical realism or (holistic) conceptualism a second-order logic with nominalized predicates should contain all of the theorems of "standard" second-order predicate logic as a proper part.

This means in particular that we should retain all of the theorems of classical propositional logic, and that all instances of the comprehension principle (CP) of "standard" second-order logic—i.e., instances in which abstract object terms do not occur—should be provable.

Initially, we will assume standard first-order predicate logic with identity as well; but, as we will see, it may be appropriate to assume "free" first-order predicate logic instead.

Note: With λ-abstracts, the comprehension principle can be stated in a stronger and more natural form as an identity:

(CP$^*_\lambda$) $(\exists F)([\lambda x_1 ... x_n \varphi] = F)$.

Because it stated as an identity, (CP$^*_\lambda$) is stronger than (CP). That is, (CP$^*_\lambda$) implies (CP), but it is not implied by (CP).

One of the rules for the new λ-operator is the rule of **λ-conversion,** (λ-Conv*):

$$[\lambda x_1 ... x_n \varphi](a_1, ..., a_n) \leftrightarrow \varphi[a_1/x_1, ..., a_n/x_n]$$

Note: With **standard** first-order predicate logic, we have

$$(\exists y)(F = y)$$

as provable for every nominalized predicate F, and therefore also for λ-abstracts as well:

$$(\exists y)([\lambda x_1 ... x_n \varphi] = y).$$

Another consequence is that the first-order principle of universal instantiation now also applies to nominalized predicates as abstract object terms:

(UI$^*_1$) $(\forall x)\varphi \rightarrow \varphi[F/x]$.

Note: It would be ideal if the comprehension principle (CP^*_λ) can be assumed for all formulas φ, including those in which nominalized predicates occur as abstract object terms.

But if the logic is not "free of existential presuppositions" for object terms, such an unrestricted second-order logic—which is similar to the system of Gottlob Frege's *Grundgesetze*—is subject to Russell's paradox of predication, and therefore cannot be assumed as a consistent principle within the framework as so far described.

Consider, e.g., the formula that represents the Russell property of *being-identical-to-*
a-property-that-is-not-predicable-of-itself. As a λ-abstract this can be formalized as

$$[\lambda x(\exists G)(x = G \wedge \neg G(x))].$$

Putting this λ-abstract into the unrestricted comprehension principle (CP^*_λ), we have

$$(\exists F)([\lambda x(\exists G)(x = G \wedge \neg G(x))] = F)$$

as provable, and from this a contradiction follows, namely that the Russell property both is and is not a property of itself. This is what is known as Russell's paradox of predication.

Russell himself later turned to his theory of ramified types as a way to avoid the contradiction. But it was later pointed out that a theory of simple types sufficed, at least for the so-called logical paradoxes such as Russell's.

The idea of a hierarchy of types based on the fundamental asymmetry of subject and predicate is fundamentally correct. But the idea can be simplified even further within a strictly second-order predicate logic with nominalized predicates as abstract object terms, a logic that is both consistent and equivalent to the simple theory of types.

4. Homogeneous Stratification

Russell's paradox of predication was generated by simply extending the consistent framework of standard second-order predicate logic to include nominalized predicates as abstract terms. The implications of this result for logical realism as a modern form of Platonism were profound.

How could mathematics be explained both ontologically and epistemologically if the formal theory of predication represented by second-order predicate logic with nominalized predicates as abstract terms is inconsistent?

This was the situation that confronted Russell in his 1903 *Principles of Mathematics*. In fact, the form of logical realism that Russell had in mind in 1903 was essentially the second-order predicate logic with nominalized predicates as abstract terms that we have so far described.

Beginning in 1903, and for some years afterwards, Russell tried to resolve his paradox in many different ways. It was not until 1908 that he settled on his theory of logical types.

What the theory of simple logical types does is divide the predicate expressions and their corresponding abstract object terms into a hierarchy of different types, and then it imposes a grammatical constraint that nominalized predicates can occur as argument- or subject-expressions only of predicates of higher types.

These purely grammatical constraints excludes from the theory expressions of the form $F(F)$, as well as their negations, $\neg F(F)$, which are just the types of expressions needed to generate Russell's paradox. This was all that Russell needed to avoid his paradox of predication.

These grammatical constraints are undesirable because they exclude as meaningless many expressions that are not only grammatically correct in natural language but also intuitively meaningful, and sometimes even true, such as the statement that the property of being red is not itself red, or that the property of being an abstract entity is itself an abstract entity.

Fortunately, it turns out, the logical insights behind these constraints can be retained while mitigating the constraints themselves. In particular, we need only impose a constraint on λ-abstracts, namely that they be restricted to those that are **homogeneously stratified** in a metalinguistic sense, and not as a grammatical distinction between types of predicates in the object language.

Now, what it means to say that a formula or λ-abstract φ is **homogeneously stratified** (**h-stratified**) is that natural numbers (representing levels) can be assigned to the terms and predicate expressions occurring in φ so that all of the terms occurring as arguments of a predicate are assigned the same number (level) with the predicate itself being assigned one number (level) higher. (See cited website for detailed definition.)

The one constraint needed to retain a consistent version of Russell's earlier 1903 logical realism is that all **λ-abstracts** must be homogeneously stratified in the metalinguistic sense just indicated. This means that formulas of the form $F(F)$ and $\neg F(F)$ are still grammatically meaningful even though they are not h-stratified.

But note that the complex predicate that is involved in Russell's paradox, namely,

$$[\lambda x(\exists G)(x = G \wedge \neg G(x))],$$

is not h-stratified, because, x and G must be assigned the same number (level) for their occurrence in $x = G$, whereas G must also be assigned the successor of what x is assigned for their occurrence in $\neg G(x)$.

The comprehension principle (CP^*_λ) and the second-order logic of the previous section can be retained in its entirety, with the one restriction that the λ-abstracts that occur in the formulas of this logic must all be h-stratified. Because of this one restriction we will refer to the system as λHST^*.

Finally, let us note that not only is Russell's paradox blocked in λHST^*, but so are other logical paradoxes as well. In fact, as we have shown elsewhere, λHST^* is consistent relative to Zermelo set theory and equiconsistent with the simple theory of logical types. Also, if we were to add to λHST^* the following axiom of extensionality:

(Ext*) $(\forall x_1)...(\forall x_n)[\varphi \leftrightarrow \psi] \rightarrow$
$[\lambda x_1...x_n\varphi] = [\lambda x_1...x_n\psi]$,

then the result is equiconsistent with the set theory known as NFU (New foundations with Urelements) as well.

Metatheorem: λHST^* is consistent relative to Zermelo set theory; and it is equiconsistent with the theory of simple logical types. λHST^*+ (Ext*) is equiconsistent with the set theory NFU.

5. Frege's Logic Reconstructed

Gottlob Frege's form of logical realism as described in his *Grundgesetze* was also a second-order predicate logic with nominalized predicates as abstract singular terms, and it too was subject to Russell's paradox.

Frege also had a hierarchy of universals implicit in his logic, except that he assumed that all higher levels of his hierarchy beyond the second could be reflected downward into the second level, which in turn was reflected in the first level of objects, which is implicitly the situation that is represented in λHST^*. In this respect, λHST^* can also be used as a consistent reconstruction of Frege's form of logical realism.

Unlike Russell, however, Frege did not assume that what a nominalized predicate denotes as an abstract singular term is the same universal that the predicate stands for in its predicative role. In particular, Frege's universals have *an unsaturated nature*, and this unsaturated nature precludes them from being objects.

For this reason, Frege's universals cannot be what nominalized predicates denote as abstract singular terms. In other words, in Frege's ontology what a predicate stands for in it predicative role is not what a nominalized predicate denotes as an abstract singular term.

Why then have nominalized predicates at all? In Frege's ontology it was not just to explain an important feature of natural language. Rather, it was a matter of **"how we are to conceive of logical objects," and numbers in particular**.

"By what means," Frege asked, "are we justified in recognizing numbers as objects?" The answer, for Frege, was that we apprehend logical objects as the extensions of properties and relations, and it is through the process of nominalization that we are able to achieve this. Here, it is the logical notion of a class as the extension of a property, or concept, that is involved, and not the mathematical notion of a set.

It was Frege's commitment to an extensional logic that led him to take classes as the objects denoted by nominalized predicates.

A class, after all, is the extension of a predicate as well as of the property or concept that the predicate stands for, and it was in terms of classes and classes of classes that Frege proposed to construct the natural numbers. That, in fact, is the basis of his ontological logicism.

Now given Frege's commitment to an extensional logic, it is not just λHST* that we should take as a consistent reconstruction of his logic, but λHST*+ (Ext*), which, as already noted, is equiconsistent with the set theory NFU and consistent relative to Zermelo set theory.

The extensionality axiom, (Ext*), incidentally, is one direction of Frege's well-known Axiom V, which was critical to the way Russell's paradox was proved in Frege's logic. This direction was called Basic Law Vb. The other direction, Basic Law Va, is actually an instance of Leibniz's law in λHST*. That is, by Leibniz's law, Frege's Basic Law Va,

$F = G \rightarrow$
$(\forall x_1)...(\forall x_n)[F(x_1, ...x_{,n}) \leftrightarrow G(x_1, ...x_{,n})]$

is provable in λHST*, independently of (Ext*), which was Frege's Basic Law Vb.

Note that because λHST*+ (Ext*) is consistent (relative to Zermelo set theory), it is not Frege's Basic Law V that is the problem. Rather it was because Frege had heterogeneous, and not just homogeneous, relations in his logic, including heterogenous relations between universals and objects, such as that of predication, and these were included as part of the reflection downward of his hierarchy.

The hierarchy consistently represented in λHST*, on the other hand, consists only of homogeneous relations.

The representation of heterogenous relations can be retained, however, by turning to an alternative reconstruction of Frege's logic that is closely related to λHST*.

This alternative involves replacing the standard first-order logic that is part of λHST* with a logic that is free of existential presuppositions regarding singular terms, including nominalized predicates such as that corresponding to the complex predicate involved in Russell's paradox.

In an appendix to his *Grundgesetze* Frege considered resolving Russell's paradox by allowing that "there are cases where an unexceptional concept has no extension". Here, by an "unexceptional concept" Frege had the rather exceptional Russell property (or concept) in mind.

But allowing that the Russell property has no extension in Frege's logic requires allowing the nominalized form of the Russell predicate to denote nothing. That is, it requires a shift from standard first-order logic to a logic free of existential presuppositions regarding singular terms, including especially nominalized predicates.

In fact, this strategy works. By adopting a free first-order logic and yet retaining the unrestricted comprehension principle (CP^*_λ), all that follows by the argument for Russell's paradox is that there is no *object* corresponding to the Russell property, i.e.,

$$\neg(\exists y)([\lambda x(\exists G)(x = G \wedge \neg G(x))] = y)$$

is provable, even though, by (CP^*_λ), the Russell property "exists" as a *property* (or concept).

In other words, both

$$(\exists F)([\lambda x(\exists G)(x = G \wedge \neg G(x))] = F)$$

and

$$\neg(\exists y)([\lambda x(\exists G)(x = G \wedge \neg G(x))] = y)$$

are provable on this approach.

Revised in this way, our original second-order logic with nominalized predicates can be easily shown to be consistent. But that is because, without any further assumptions, we can no longer prove that any property or relation has an extension. That is, because the logic is free of existential presuppositions, all nominalized predicates might be denotationless, a position that a nominalist might well adopt.

But in Frege's ontological logicism some properties and relations must have extensions, and, in fact, it is appropriate to assume that all of the properties and relations that can be represented in λHST* have extensions in this alternative logic. That in fact is exactly what we allow in our alternative reconstruction of Frege's logic.

The added assumption can be stipulated in the form of an axiom schema in which all predicate quantifiers in the formula in question refer only to those properties and relations (or concepts) that have objects corresponding to them, in which case we say that the formula is *bound to objects*. The axiom schema is then given as follows:

$$(\exists/\text{HSCP}^*_\lambda) \qquad (\exists y)(a_1 = y) \wedge ... \wedge (\exists y)(a_k = y) \rightarrow (\exists y)([\lambda x_1...x_n\varphi] = y),$$

where, (1) $[\lambda x_1...x_n\varphi]$ is h-stratified, (2) φ is bound to objects, (3) y is an object variable not occurring in φ, and (4) $a_1, ..., a_k$ are all of the object or predicate variables or nonlogical constants occurring free in $[\lambda x_1...x_n\varphi]$.

Because of it close similarity to our first reconstructed system, λHST*, we will refer to this alternative logic as $\mathbf{HST}^*_{\boldsymbol{\lambda}}$.

Theorem: HST$^*_\lambda$ is consistent if and only if λHST* is consistent, and therefore if and only if the theory of simple types is consistent. All three systems are consistent relative to Zermelo set theory.

Finally, let us note that although HST^*_λ + (Ext*) can be taken as a reconstruction of Frege's logic and ontology, it cannot also be taken as a reconstruction of Russell's early (1903) ontology, with or without the extensionality axiom, (Ext*).

This is because Russell rejected Frege's notion of unsaturatedness and took nominalized predicates to denote as singular terms the same concepts and relations they stand for as predicates. In other words, unlike Frege, Russell cannot allow that some predicates stand for properties and relations (or concepts), but that as singular terms their nominalizations denote nothing.

Of course, we do have the system λHST^*, which can be taken as an unproblematic reconstruction of Russell's early ontological framework.

6. Conceptual Intensional Realism

In conceptualism, predicable concepts are cognitive capacities that underlie our rule-following abilities in the use of the predicate expressions of natural language.

Note: As capacities that can be exercised by different persons at the same time, as well as by the same person at different times, concepts cannot be objects, e.g., ideas or mental images as particular mental occurrences.

In other words, as intersubjectively realizable cognitive capacities, concepts are objective and not merely subjective entities.

Moreover, as essential components of predication in language and thought, concepts as cognitive capacities have an unsaturated nature, and it is this unsaturated nature that is the basis of predication in language and thought.

In particular, it is the exercise of a predicable concept in a speech or mental act that informs that act with a predicable nature, a nature by means of which we characterize and relate objects in various ways.

The unsaturatedness of a concept as a cognitive capacity is not the same as the unsaturatedness of a function in Frege's ontology.

For Frege, a property or relation is really a function from objects to truth values, and it is part of the nature of every function, according to Frege, even those from numbers to numbers, to be unsaturated. In Frege's ontology, in other words, predication is a type of functionality.

In conceptualism it is predication that is more fundamental than functionality. We understand what it means to say that a function assigns truth values to objects, after all, only by knowing what it means to predicate concepts, or properties and relations, of objects.

The unsaturated nature of a concept is not that of a function, but of a cognitive capacity that could be exercised by different people at the same time as well as by the same person at different times, and in fact some concepts might not even be exercised ever at all. When such a capacity is exercised, however, what results is not a truth value, but a mental event, i.e., a mental act, and if expressed overtly in language, a speech-act event as well.

Now if predicable concepts are unsaturated cognitive structures, then why should conceptualism have a logic of nominalized predicates as abstract singular terms as well as a logic of predication?

One reason is the same as it was for Frege: namely, to account for the ontology of the natural numbers as logical objects.

Another reason is to explain the significance of nominalized predicates in natural language, including especially complex forms of predication containing infinitives, gerunds, and other abstract nouns. Such an account will involve positing abstract intensional objects as the denotata of nominalized predicates, which, unlike Frege's commitment to an extensional logic, will commit conceptualism to an intensional logic.

Instead of denoting the extensions of concepts, in other words, nominalized predicates in conceptualism are assumed to denote the intensional contents of concepts "object"-ified as abstract intensional objects. This is why we refer to this extension of conceptualism as conceptual intensional realism.

Note: By the intensional content of a predicable concept we understand the truth conditions determined by the different possible applications of that concept, i.e., the conditions under which objects can be said to fall under the concept in any possible context of use, including fictional contexts.

Of course, there are some predicable concepts, such as that represented by the Russell predicate,

$$[\lambda x(\exists G)(x = G \wedge \neg G(x))],$$

that determine truth conditions corresponding to which, logically, there can be no corresponding abstract object, on pain otherwise of contradiction.

This does not mean that such a predicable concept does not determine truth conditions and therefore does not have intensional content. Rather, it means only that such a content cannot be "object"-ified, i.e., there cannot be an abstract object corresponding to the content of that concept, the way there are for the contents of other concepts.

The real lesson of Russell's paradox is that some rather exceptionable, "impredicatively" constructed concepts determine truth conditions that logically cannot be "object"-ified, whereas most predicable concepts are unexceptionable in this way.

In conceptual Platonism, incidentally, the abstract object corresponding to the intensional content of a predicable concept is a Platonic Form, which traditionally has been called a property or relation — a terminology that we can allow as well in conceptual realism so long as we do not confuse these properties and relations with the natural properties and relations of conceptual natural realism.

There is an important ontological difference between conceptual Platonism and conceptual intensional realism, however, despite the similarity of both to logical realism.

First, note that unlike logical realism conceptual Platonism is *an indirect* and *not a direct* Platonism.

That is, in conceptual Platonism (and conceptual realism), but not in logical realism, abstract objects are cognized only indirectly through the concepts whose correlates they are. This means that our representation of abstract objects is seen as a reflexive abstraction corresponding to the process of nominalization.

In other words, according to **conceptual Platonism**, even though abstract objects exist in a realm that transcends space, time and causality, and in that sense preexist the evolution of consciousness and the cognitive capacities that we exercise in thought and our use of language, nevertheless, from an epistemological point of view, no such entity can be cognized otherwise than as the correlate of a concept, i.e., as an abstract intensional object corresponding to the truth conditions determined by that concept.

In **conceptual intensional realism**, on the other hand, all abstract objects, despite having a certain amount of autonomy, have being only as products of language and culture.

Abstract do not preexist the evolution of consciousness and the cognitive capacities that we exercise in language, in other words, but are evolutionary products of language and culture, and therefore depend ontologically on language and culture for their "existence," or being.

Of course, abstract objects, especially numbers, are also an essential part of the means whereby further cultural development becomes possible. Nevertheless, as cultural products, the "existence", or being, of abstract objects is primarily the result, and development of, the kind of reflexive abstraction that is represented by the process of nominalization.

It was through the institutionalization of this process that abstract objects achieved a certain autonomy and, in time, became reified as objects. Abstract objects do not exist in a Platonic realm outside of space, time, and causality, on this interpretation, but are in fact the result, in effect, of an ontological projection inherent in the development and institutionalization in language of the process of nominalization.

The fundamental insight into the nature of abstract objects according to conceptual intensional realism is that we are able to grasp and have knowledge of such objects as the "object"-ified truth conditions of the concepts whose contents they are, i.e., as the object correlates of those concepts.

This "object"-ification of truth conditions is realized, moreover, through a kind of reflexive abstraction in which we attempt to represent what is not an object—in particular an unsaturated cognitive structure underlying our use of a predicate expression—as if it were an object. In language this reflexive abstraction is institutionalized in the rule-based linguistic process of nominalization.